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BOUNDS FOR THE MINIMUM VERTEX-CONNECTIVITY ENERGY OF A GRAPH

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ABSTRACT

In this paper, we put forward the new variant of graph energy namely, the minimum vertex connectivity energy of a graph. Further, we have obtained some bounds for this newly introduced parameter.

KEYWORDS: vertex connectivity, Energy, Minimum vertex-connectivity energy.
Subject Classification: 05C50.

1. INTRODUCTION

Let $G = (V, E)$ be any connected graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. For any integer x, $\lceil \frac{x}{2} \rceil$ is the largest integer greater than or equal to x. A subset C of a vertex set V is said to be vertex-connectivity set if the removal of vertices in C results in a disconnected graph. The minumum cardinality among such a set is considered for our study. For undefined terminologies we refer the reader to [7].

The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. This quantity, introduced almost 30 years ago [8] and having a clear connection to chemical problems [16], has in newer times attracted much attention of mathematicians and mathematical chemists [1, 4, 5, 8, 9, 10, 11, 12, 15, 16, 17, 19].

The vertex connectivity matrix is defined as follows.

Definition 1. Let C be any minimum vertex-connectivity set of G. The minimum covering matrix of G is the $n \times n$ matrix $A_c(G) = (a_{i,j})$, where

$$a_{ij} = \begin{cases} 1, & if \ \ v_i v_j \ \in \ E; \\ 1, & if \ \ i \ = \ j \ and \ \ v_i \ \in \ C; \\ 0, & otherwise. \end{cases}$$

The characteristic polynomial of $A_c(G)$ is denoted by

$$f_n(G, \lambda) := det(\lambda I - A_c(G))$$

The minimum vertex-connectivity eigenvalues of a graph G are the eigenvalues of $A_c(G)$. Since $A_c(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The vertex-connectivity energy of G is then defined as

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In this paper, some new bounds for the vertex-connectivity energy $E_c(G)$ of a graph G are presented.

2. MAIN RESULTS

For the sake of completeness, we mention below some results which are important throughout the paper.

Lemma 1 *[1] If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_c(G)$, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 = 2m + |C|. \quad (1)$$

Theorem 1 *[14] Suppose a_i and b_i, $1 \le i \le n$ are positive real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_ib_i\right)^2 \tag{2}$$

where $M_1 = max_{1\le i\le n}(a_i)$; $M_2 = max_{1\le i\le n}(b_i)$; $m_1 = min_{1\le i\le n}(a_i)$ and $m_2 = min_{1\le i\le n}(b_i)$

Theorem 2 *[13] Let a_i and b_i, $1 \le i \le n$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_ib_i\right)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2 \tag{3}$$

where M_i and m_i are defined similarly to Theorem 1.

Theorem 3 *[2] Suppose a_i and b_i, $1 \le i \le n$ are positive real numbers, then*

$$\left|n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \le \alpha(n)(A-a)(B-b) \tag{4}$$

where a, b, A and B are real constants, that for each i, $1 \le i \le n$, $a \le a_i \le A$ and $b \le b_i \le B$. Further, $\alpha(n) = n\lfloor\frac{n}{2}\rfloor\left(1 - \frac{1}{n}\lfloor\frac{n}{2}\rfloor\right)$.

Theorem 4 *[6] Let a_i and b_i, $1 \le i \le n$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \le (r+R)\left(\sum_{i=1}^{n} a_ib_i\right) \tag{5}$$

where r and R are real constants, so that for each i, $1 \le i \le n$, holds, $ra_i \le b_i \le Ra_i$.

3. BOUNDS ON THE MINIMUM COVERING ENERGY OF A GRAPH

In this section, a variety of lower bounds for the vertex-connectivity energy of a graph are presented.

Theorem 5 *Suppose zero is not an eigenvalue of $A_c(G)$. Then*

$$E_c(G) \ge \frac{2\sqrt{\lambda_1\lambda_n}\sqrt{(2m+|C|)n}}{\lambda_1+\lambda_n}. \tag{6}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i's$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_c(G)$. We assume that $a_i = |\lambda_i|$ and $b_i = 1$, which by Theorem 1 implies

$$\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 \le \frac{1}{4}\left(\sqrt{\frac{\lambda_n}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_n}}\right)^2 \left(\sum_{i=1}^{n} |\lambda_i|\right)^2$$

$$(2m+|C|)n \le \frac{1}{4}\left(\frac{(\lambda_1+\lambda_n)^2}{\lambda_1\lambda_n}\right)(E_c(G))^2$$

$$E_c(G) \ge \frac{2\sqrt{\lambda_1\lambda_n}\sqrt{(2m+|C|)n}}{\lambda_1+\lambda_n},$$

as desired.

Theorem 6 *Let G be a graph of order n and size m, then*

$$E_c(G) \ge \sqrt{(2m+|C|)n - \frac{n^2}{4}(\lambda_n - \lambda_1)^2} \tag{7}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i's$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_c(G)$. We assume that $a_i = 1$ and $b_i = |\lambda_i|$, which by Theorem 2 implies

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \frac{n^2}{4}(\lambda_n - \lambda_1)^2$$

$$(2m+|C|)n - (E_c(G))^2 \le \frac{n^2}{4}(\lambda_n - \lambda_1)^2$$

$$E_c(G) \ge \sqrt{(2m+|C|)n - \frac{n^2}{4}(\lambda_n - \lambda_1)^2},$$

as asserted.

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Theorem 7 *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_C(G)$. Then*

$$E_c(G) \geq \sqrt{2mn + n|C| - \alpha(n)(|\lambda_1| - |\lambda_n|)^2} \tag{8}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_c(G)$. We assume that $a_i = |\lambda_i| = b_i$, $a = |\lambda_n| = b$ and $A = |\lambda_1| = b$, which by Theorem 3 implies

$$|n\sum_{i=1}^{n} |\lambda_i|^2 - (\sum_{i=1}^{n} |\lambda_i|)^2| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \tag{9}$$

Since, $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2m + |C|$, the above inequality becomes,

$$(2m + |C|)n - E(G)^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2,$$

wherefrom (8) follows.

Corollary 8 *Since $\alpha(n) \leq \frac{n^2}{4}$, then according to (8), we have*

$$E_c(G) \geq \sqrt{2mn + n|C| - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$$
$$\geq \sqrt{2mn + n|C| - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}.$$

This means that inequality (8) is stronger of inequality (7).

Theorem 9 *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_C(G)$. Then*

$$E_c(G) \geq \frac{|\lambda_1||\lambda_n|n + 2m + |C|}{|\lambda_1| + |\lambda_n|} \tag{10}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i's$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_c(G)$. We assume that $b_i = |\lambda_i|$, $a_i = 1$ $r = |\lambda_n|$ and $R = |\lambda_1|$, which by Theorem 4 implies

$$\sum_{i=n}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|. \tag{11}$$

Since, $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2m + |C|$, from (11), inequality (10) directly follows from Theorem 4.

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